JOURNAL OF APPROXIMATION THEORY 76, 123–130 (1994)

New Bounds on the Zeros of Spline Functions

T. N. T. GOODMAN

Department of Mathematics and Computer Science, The University of Dundee, Dundee DD1 4HN, Scotland

Communicated by C. de Boor

Received October 24, 1991; accepted in revised form August 21, 1992

We show that, subject to a certain condition, the number of zeros of a spline function is bounded by the number of strong sign changes in its sequence of B-spline coefficients. By writing a general spline function as a sum of functions which satisfy the given condition, we can deduce known bounds on zeros and sign changes and can show that the number of zeros of any spline function is bounded by the number of weak sign changes in its sequence of B-spline coefficients, where the zero count is stronger than that previously used. © 1994 Academic Press, Inc.

1. BACKGROUND

The relationship between a spline function and the sequence of its B-spline coefficients has been much studied [1, 7, Chap. 4] and it seems surprising that there is anything simple or useful left to say in this regard. However, we give below a simple example for which the known results do not imply the number of zeros which seems intuitively clear, and examples such as this impelled us to prove stronger results on the zeros and sign changes of spline functions.

First we give some definitions. For a real sequence $\mathbf{a}=(a_i)_0^k$, $S^-(\mathbf{a})$ and $S^+(\mathbf{a})$ denote respectively the minimum and maximum number of sign changes in $\mathbf{a}$ gained by assigning signs to the zero entries. For a real-valued function f on an interval I, we let $V(f)$ denote the number of its changes of sign, i.e.,

$$V(f):=\sup S^-(f(t_0), ..., f(t_k)),$$

where the supremum is taken over all sequences $t_0<\cdots<t_k$ in I, for all k.

Now take integers $m, n\geqslant 0$ and a non-decreasing sequence $\mathbf{t}=(t_i)_0^{m+n+1}$ with $t_i<t_{i+n+1}$, $i=0, ..., m$. For $i=0, ..., m$, let N_i denote the B-spline of degree n with knots $t_i, ..., t_{i+n+1}$. (The normalisation is immaterial.) For constants $a_0, ..., a_n$ we let

$$f(x)=\sum_{i=0}^{m} a_i N_i(x), \qquad t_0<x<t_{m+n+1}, \qquad \mathbf{a}=(a_i)_0^m. \tag{1}$$

0021-9045/94 $6.00

We denote by $Z(f)$ the number of its zeros, counted with multiplicity as in [7, Chap. 4]. (This depends on $\mathbf{t}$ and is described in Section 3 where it is compared to our stronger zero count.) It is known [7, Chap. 4] that

$$V(f) \leqslant S^-(\mathbf{a}), \tag{2}$$

$$Z(f) \leqslant m. \tag{3}$$

A stronger form of (2) is given in [2], while (3) can be strengthened to

$$Z(f) \leqslant S^+(\mathbf{a}), \tag{4}$$

see [4, Theorem 1].

Now take $0 \leqslant i_0 < i_1 < \cdots < i_r \leqslant m$ and let

$$f = \sum_{j=0}^{r} b_j N_{i_j}.$$

Then de Boor [2] and de Boor and DeVore [3] have shown the following, which is a stronger version of the Schoenberg–Whitney theorem [6].

If f vanishes at points $y_0 \leqslant y_1 \leqslant \cdots \leqslant y_r$ with

$$t_{i_j} < y_j < t_{i_j+n+1}, \qquad j = 0, ..., r, \tag{5}$$

then $f = 0$. (Here, as usual, coincidence of points y_i denotes vanishing of derivatives.)

We now give the promised example. Let $n = 2$, $m = 6$, $\mathbf{t} = (t_i)_0^9$ be strictly increasing, and as in (1) define

$$f(x) = a_0 N_0(x) + a_2 N_2(x) + a_4 N_4(x) + a_6 N_6(x), \qquad t_0 < x < t_9,$$

where $a_0 > 0$, $a_2 > 0$, $a_4 < 0$, $a_6 < 0$. Here (2) gives $V(f) \leqslant 1$, (3) gives $Z(f) \leqslant 6$, (4) gives $Z(f) \leqslant 5$, and (5) gives $Z(f) \leqslant 3$. None of these results tell us that $Z(f) \leqslant 1$. In the next section we prove the following result which, as we shall see, implies (3), (4), and (5) and, in particular, gives $Z(f) \leqslant 1$ in the above example.

2. The Main Result

THEOREM 1. *Suppose that f is given by* (1) *and that for every x in (t_0, t_{m+n+1}), there is some i with $a_i \neq 0$ and $t_i < x < t_{i+n+1}$. Then*

$$Z(f) \leqslant S^-(\mathbf{a}). \tag{6}$$

Before proving this, we clarify the meaning of $Z(f)$ in this case. By the local linear independence of B-splines, the condition of Theorem 1 implies that f cannot vanish on any non-trivial interval in (t_0, t_{m+n+1}). Now suppose that x is a knot of multiplicity $s \geqslant 1$ in (t_0, t_{m+n+1}), so that $x = t_j$, where $t_{j-1} < t_j = \cdots = t_{j+s-1} < t_{j+s}$. Then $t_i < x < t_{i+n+1}$ can hold only for $i = j+s-n-1, ..., j-1$. The condition of the theorem then shows that $s \leqslant n$ and thus there are no knots of multiplicity $n+1$ and f is continuous. If $f^{(k)}(x) = 0$, $k = 0, ..., n-s$, then applying (5) to the function $\sum_{i=j+s-n-1}^{j-1} a_i N_i$, shows that $a_i = 0$ for $i = j+s-n-1, ..., j-1$, which contradicts the condition of Theorem 1. Thus at a knot of multiplicity s, f can have a zero of multiplicity at most $n-s$. To sum up, any zero of f is a point y, where for some $l \geqslant 1$, $f^{(j)}(y) = 0$, $j = 0, ..., l-1$, $f^{(l)}$ is continuous at y and $f^{(l)}(y) \neq 0$. The zero count $Z(f)$ is in this case gained by simply adding the number of zeros y with corresponding multiplicities l. This avoids the more general zero count of [7] which has to deal with interval zeros and discontinuities, and is discussed in Section 3.

The proof of Theorem 1 depends on the following, which is Lemma 3 of [4]. The general approach was initiated in [5].

LEMMA. *Suppose that f is given by* (1) *and take τ with $t_j < \tau < t_{j+n+1}$ for some j with $a_j \neq 0$. Let $\tilde{\mathbf{t}} = (\tilde{t}_i)_0^{m+n+2}$ denote the non-decreasing sequence gained by including τ in $\mathbf{t}$. For $i = 0, ..., m+1$, we assume $\tilde{t} < \tilde{t}_{i+n+1}$ and let $\tilde{N}_i$ denote the B-spline of degree n with knots $\tilde{t}_i, ..., \tilde{t}_{i+n+1}$. If*

$$f = \sum_{i=0}^{m+1} \tilde{a}_i \tilde{N}_i, \qquad \tilde{\mathbf{a}} = (\tilde{a}_i)_0^{m+1},$$

then

$$S^+(\tilde{\mathbf{a}}) \leqslant S^+(\mathbf{a}).$$

Proof of Theorem 1. Let $f = f_1 + \cdots + f_r$, where for $j = 1, ..., r$,

$$f_j = \sum_{i=l_j}^{m_j} a_i N_i, \qquad a_i \neq 0 \qquad \text{for} \quad l_j \leqslant i \leqslant m_j,$$

and

$$l_{j+1} \geqslant m_j + 2, \qquad j = 1, ..., r-1.$$

Let $\alpha = t_{l_2}$, $\beta = t_{m_1+n+1}$. Since $a_i \neq 0$ implies $i \leqslant m_1$ or $i \geqslant l_2$, the assumption of Theorem 1 implies $t_i < \alpha < t_{i+n+1}$, for some $i \leqslant m_1$ and hence $\alpha < t_{m_1+n+1} = \beta$. Letting $k = l_2 - m_1 - 1$, we construct a new sequence $\tilde{\mathbf{t}} = (\tilde{t}_i)_0^{m+n+k+1}$ by inserting k distinct points into $\mathbf{t}$, where these points are in (α, β) and distinct from the elements of $\mathbf{t}$. Let $\tilde{N}_i$ denote the B-spline with knots $\tilde{t}_i, ..., \tilde{t}_{i+n+1}$, and let $f = \sum_0^{m+k} \tilde{a}_i \tilde{N}_i$. Then $f_1 = \sum_{l_1}^{m_1+k} \tilde{a}_i \tilde{N}_i$ with $a_{m_1} \tilde{a}_{m_1+k} > 0$, and $f_2 = \sum_{l_2}^{m_2+k} \tilde{a}_i \tilde{N}_i$, with $a_{l_2} \tilde{a}_{l_2} > 0$. By the lemma,

$$S^+(\tilde{a}_{l_1}, ..., \tilde{a}_{m_1+k}) \leqslant S^+(a_{l_1}, ..., a_{m_1}) = S^-(a_{l_1}, ..., a_{m_1})$$

and similarly

$$S^+(\tilde{a}_{l_1}, ..., \tilde{a}_{m_2+k}) \leqslant S^-(a_{l_2}, ..., a_{m_2}).$$

Since $m_1 + k = l_2 - 1$,

$$S^+(\tilde{a}_{l_1}, ..., \tilde{a}_{m_2+k}) \leqslant S^-(a_{l_1}, ..., a_{m_2}).$$

Continuing in this way we can construct a non-decreasing sequence $\hat{\mathbf{t}} = (\hat{t}_i)_0^{m+n+1+l}$, some $l \geqslant 1$, with corresponding B-splines $\hat{N}_i$, $i = 0, ..., m+l$, so that $f = \sum_0^{m+l} \hat{a}_i \hat{N}_i$ and

$$S^+(\hat{a}_0, ..., \hat{a}_{m+l}) \leqslant S^-(a_0, ..., a_m).$$

The result now follows from (4). ∎

If a spline function f as in (2) does not satisfy the condition of Theorem 1, then we can simply split it into pieces which do satisfy the condition, as follows. It is easily seen that for some $p \geqslant 1$ we can choose points

$$t_0 \leqslant b_1 < c_1 \leqslant b_2 < c_2 \leqslant \cdots \leqslant b_p < c_p \leqslant t_{m+n+1}, \tag{7}$$

so that for $i = 1, ..., p$, $f \mid (b_i, c_i)$ satisfies the condition of Theorem 1, and f vanishes outside $\bigcup \{[b_i, c_i] : i = 1, ..., p\}$. Thus we can deduce zero and sign change properties of f by applying the theorem to $f \mid (b_i, c_i)$, $i = 1, ..., p$. As an example let $n = 2$, $m = 9$, $\mathbf{t} = (t_i)_0^{12}$ be strictly increasing, and define

$$\begin{aligned} f(x) = a_0 N_0(x) + a_2 N_2(x) + a_4 N_4(x) \\ + a_6 N_6(x) + a_9 N_9(x), \qquad t_0 < x < t_{12}, \end{aligned} \tag{8}$$

where $a_0 > 0$, $a_2 > 0$, $a_4 < 0$, $a_6 < 0$, $a_9 > 0$. Here $b_1 = t_0$, $c_1 = b_2 = t_9$, $c_2 = t_{12}$. On (b_1, c_1) this reduces to our previous example and there is a single simple zero. On (b_2, c_2), $f(x) = a_9 N_9(x)$ and does not vanish, while at t_9 there is a double zero at which f changes sign.

By splitting up the function in the above manner and applying Theorem 1 to each piece, it is easy to deduce (3) and (5). In the next section we deduce a stronger version of (4).

3. A General Zero Bound

We first strengthen the zero count in [7, Chap. 4]. We suppose that f is given by (1) and is not identically zero and consider different types of zeros.

Isolated Zeros

1. *Interior point.* Take x in (t_0, t_{m+n+1}) and suppose that x has multiplicity $s \geqslant 0$ in $\mathbf{t}$. Suppose that for some l, r, $0 \leqslant l \leqslant n$, $0 \leqslant r \leqslant n$,

$$f(x^-) = f'(x^-) = \cdots = f^{(l-1)}(x^-) = 0 \neq f^{(l)}(x^-), \tag{9}$$

$$f(x^+) = f'(x^+) = \cdots = f^{(r-1)}(x^+) = 0 \neq f^{(r)}(x^+). \tag{10}$$

Since f is C^{n-s} at x, we have $l \leqslant n-s$ if and only if $r \leqslant n-s$ and in this case we have $l = r$ and define

$$\alpha(x) = l = r. \tag{11}$$

We have seen that $l \geqslant n-s+1$ if and only if $r \geqslant n-s+1$ and in this case we define

$$\alpha(x) = l + r + s - n - 1.$$

Now we define

$$z(x) = \begin{cases} \alpha(x)+1, & \text{if } \alpha(x) \text{ is even and } f \text{ changes sign across } x, \\ \alpha(x)+1, & \text{if } \alpha(x) \text{ is odd and } f \text{ does not change sign across } x, \\ \alpha(x), & \text{otherwise.} \end{cases}$$

In [7] the definition is the same but with $\alpha(x)$ replaced by $\max(l, r)$. It can easily be seen that $\alpha(x) \geqslant \max(l, r)$ and so our zero count is stronger.

2. *Left end-point.* Let $x = t_0$ and suppose that (10) holds for some r, $0 \leqslant r \leqslant n$. Let $s \geqslant 1$ denote the multiplicity of x in $\mathbf{t}$. Then $r \geqslant n-s+1$ and we define

$$z(x) = r + s - n - 1.$$

3. *Right end-point.* Let $x = t_{m+n+1}$ and suppose that (9) holds for some l, $0 \leqslant l \leqslant n$. Let $s \geqslant 1$ denote the multiplicity of x in $\mathbf{t}$. Then $l \geqslant n-s+1$ and we define

$$z(x) = l + s - n - 1.$$

(Cases 2 and 3 are not considered in [7].)

In all cases, if $z(x) \geqslant 1$ we say that x is an isolated zero of f of multiplicity $z(x)$.

Interval Zeros

1. *Interior interval.* Suppose that f vanishes on $T=(x, y)$, while for some l, r, $0 \leqslant l \leqslant n$, $0 \leqslant r \leqslant n$, (9) holds and (10) holds with x replaced by y. Let $p \geqslant 1$, $q \geqslant 1$, denote respectively the multiplicities of x and y in $\mathbf{t}$. Then we define

$$\alpha(T)=l+r+p+q-n-1+M,$$

where

$$M=|\{i : t_i \in T\}|. \tag{12}$$

We now define

$$z(T)=\begin{cases} \alpha(T)+1, & \text{if } \alpha(T) \text{ is even and } f \text{ changes sign across } T, \\ \alpha(T)+1, & \text{if } \alpha(T) \text{ is odd and } f \text{ does not change sign across } T, \\ \alpha(T), & \text{otherwise.} \end{cases}$$

In [7], $\alpha(T)$ is replaced by $n+1+M$. Since $l \geqslant n-p+1$, $r \geqslant n-q+1$, we have $\alpha(T) \geqslant n+1+M$.

2. *Left end-interval.* Suppose that f vanishes on $T=(t_0, x)$ and (10) holds for some r, $0 \leqslant r \leqslant n$. Let $q \geqslant 1$ denote the multiplicity of x in $\mathbf{t}$. Then with M as in (12) we define

$$z(T)=r+q+M.$$

(Since $r \geqslant n-q+1$, we have $z(T) \geqslant n+1+M$, which is the value in [7].)

3. *Right end-interval.* Suppose that f vanishes on $T=(x, t_{m+n+1})$ and (9) holds for some l, $0 \leqslant l \leqslant n$. Let $p \geqslant 1$ denote the multiplicity of x in $\mathbf{t}$. Then with M as in (12) we define

$$z(T)=l+p+M.$$

(As before, $z(T) \geqslant n+1+M$, which is the value in [7].) In all cases we call T an interval zero of f of multiplicity $z(T)$.

Finally we define

$$Z^+(f)=\sum z(t),$$

where the summation is taken over all isolated and interval zeros t of f.

Before stating Theorem 2, we give a simple example for which $Z^+(f) > Z(f)$. Let $n=1$, $m=3$, $t_0 < t_1 < t_2 = t_3 < t_4 < t_5$ and let $f = N_0 - N_3$. Then f has an isolated zero at $x = t_2$ with $l = r = 1$ and $\alpha(x) = 2$. Hence $Z^+(f) = 3$, whereas $Z(f) = 1$.

THEOREM 2. *If f is given by* (1) *and is not identically zero, then*

$$Z^+(f) \leqslant S^+(\mathbf{a}). \tag{13}$$

Proof. Choose points as in (7) so that for $v = 1, ..., p$, $f_v := f \mid (b_v, c_v)$ satisfies the condition of Theorem 1 and f vanishes outside $\bigcup \{[b_v, c_v] : v = 1, ..., p\}$. Suppose that for $v = 1, ..., p$,

$$f_v = \sum_{i=j_v}^{k_v} a_i N_i,$$

where $a_i \neq 0$ for $i = j_v$ and k_v. We show that

$$\alpha([c_v, b_{v+1}]) = j_{v+1} - k_v - 1, \qquad v = 1, ..., p-1, \tag{14}$$

$$z([t_0, b_1]) = j_1, \tag{15}$$

$$z([c_p, t_{m+n+1}]) = m - k_p. \tag{16}$$

Writing $c_0 = t_0$, $b_{p+1} = t_{m+n+1}$, we have

$$Z(f) = \sum_{v=1}^{p} Z(f_v) + \sum_{v=0}^{p} z([c_v, b_{v+1}])$$

and applying Theorem 1 gives

$$Z(f) \leqslant \sum_{v=1}^{p} S^+(a_{j_v}, ..., a_{k_v}) + \sum_{v=0}^{p} z([c_v, b_{v+1}]) = S^+(\mathbf{a}), \tag{17}$$

by (14), (15), and (16), which is the required result.

In fact the only case we prove is (14) for $c_v = b_{v+1}$ as (14) for $c_v < b_{v+1}$ and (15) and (16) follow similarly. Let $c_v = b_{v+1} = x$ for some v, $1 \leqslant v \leqslant p-1$. Suppose that x has multiplicity $s \geqslant 1$ in $\mathbf{t}$ and $x = t_i$, $i = \alpha, ..., \alpha + s - 1$. Then l in (9) is given by $l = \alpha - k_v - 1$, where $n + 1 - s \leqslant l \leqslant n$, and r in (10) is given by $r = j_{v+1} + n + 1 - \alpha - s$, where $n + 1 - s \leqslant r \leqslant n$. So by definition, $\alpha(x) = l + r + s - n - 1 = j_{v+1} - k_v - 1$ and (14) is established. ∎

Of course Theorem 2 implies $Z^+(f) \leqslant m$. We remark that because we replaced S^- by S^+ in (17), the inequality (13) may be weaker than could be obtained directly from Theorem 1. For example, for the function (8), Theorem 2 gives us only $Z^+(f) \leqslant 8$, while we saw in Section 2 that Theorem 1 implies that $Z^+(f) = 4$.

References

1. C. de Boor, Splines as linear combinations of B-splines: A survey, *in* "Approximation Theory II" (G. G. Lorentz, C. K. Chui, and L. L. Schumaker, Eds.), pp. 1–47, Academic Press, New York, 1976.
2. C. de Boor, Total positivity of the spline collocation matrix, *Indiana University J. Math.* **25** (1976), 541–551.
3. C. de Boor and R. DeVore, A geometric proof of total positivity for spline interpolation, *Math. Comp.* **45** (1985), 497–504.
4. T. N. T. Goodman and S. L. Lee, Interpolatory and variation diminishing properties of generalised B-splines, *Proc. Roy. Soc. Edinburgh Sect. A* **96** (1984), 249–259.
5. J. M. Lane and R. F. Riesenfeld, A geometric proof of the variation diminishing property of B-spline approximation, *J. Approx. Theory* **37** (1983), 1–4.
6. I. J. Schoenberg and A. Whitney, On Pólya frequency functions III: The positivity of translation determinants with application to the interpolation problem by spline curves, *Trans. Amer. Math. Soc.* **74** (1953), 246–259.
7. L. L. Schumaker, "Spline Functions: Basic Theory," Wiley, New York, 1981.